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Vine copulas: modelling systemic risk and enhancing higher-moment portfolio optimisation

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Abstract

Asymmetric dependence in equities markets has been shown to have detrimental effects on portfolio diversification as assets within the portfolio exhibit greater correlations during market downturns compared to market upturns. By applying the Clayton canonical vine copula (CVC) to model asymmetric dependence, we produce a measure of systemic risk across a portfolio of assets. In addition, we use the Clayton CVC to produce estimates of expected returns in an application to higher-moment portfolio optimisation and find evidence of an improvement in performance across a range of risk-adjusted return measures and the indices of acceptability.

Key words: portfolio management; behavioural finance; vine copula; Clayton copula; asymmetric dependence

JEL classification: G11, G17

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1. Introduction

Investing in diversified portfolios is a fundamental principle in modern portfolio management and mean-variance (MV) optimisation. However, it oversimplifies the challenge of portfolio construction especially when returns are non-normal (Brands and Gallagher, 2005; Benson *et al.*, 2008; Doan *et al.*, 2014) and asymmetric dependence (also known as asymmetric correlations) is shown to exist in equities markets (Longin and Solnik, 1995, 2001; Ang and Chen, 2002). Furthermore, behavioural finance studies show that managing higher moments (e.g. skewness and kurtosis) in investment portfolios is of increasing importance to investors (Kahneman and Tversky, 1979; Benson *et al.*, 2007; Qiao *et al.*, 2014) and is crucial to minimising exposure to downside risk (Chua *et al.*, 2009; Low *et al.*, 2013). As risk within the MV framework is denoted by the variance–covariance matrix (VCV) matrix, MV optimisation falls short of being able to account for the existence of asymmetric dependence within portfolios, or investor preferences for higher moments.

Ideally, risk managers would like a quantitative measure that is able to indicate when financial markets are approaching a period of turbulence to reduce risk-taking by traders. Portfolio managers want diversification on the downside and unification on the upside to maximise risk-adjusted returns on investment portfolios. We show an application of the Clayton canonical vine copula (CVC) that, by modelling asymmetric dependence across a portfolio of assets, is able to generate an estimate of aggregate asymmetric correlations that can be used as a proxy for systemic risk in an investment universe. In addition, by simulating asset returns using the Clayton CVC, we incorporate the persistence of asymmetric correlations in the estimates of expected returns. By doing so, we improve the sample inputs applied to full-scale optimisation (FSO) that leads to enhanced out-of-sample performance outcomes in higher-moment portfolio optimisation applications.

Full-scale optimisation methodology is demonstrated by Adler and Kritzman (2007) and Chua *et al.* (2009) as an optimisation technique that is able to incorporate investor preferences for higher moments (El-Hassan and Kofman, 2003; Pinnuck, 2004; Qiao *et al.*, 2014). Hagströmer *et al.* (2008) report that the magnitude of the performance improvements of FSO decreases in out-of-sample tests due to estimation errors. Thus, Hagströmer *et al.* (2008) conclude that the successful performance of FSO is dependent upon the quality of sample returns input into the optimisation process. They add that the quality of inputs is determined by the persistence of the properties of the sample return distributions during the out-of-sample period.

To reduce the impact of estimation error in FSO, application of a probability model that captures asymmetric dependence in the distribution of returns should result in enhanced portfolio performance, compared to the use of historical returns samples. Such an approach shows great promise as excessive downside correlations have been shown to exist in the US equities market (Ang and Chen, 2002) and international equities markets (Longin and Solnik, 1995, 2001). Kritzman *et al.* (2010) find that using plausible assumptions tied to economic intuition to estimate expected returns, the performance of optimised portfolios is substantially improved.

Low *et al.* (2013) apply the Clayton CVC to model asymmetric dependence in equities returns to minimise the event of extreme losses, and find improved out-of-sample performance compared to a Gaussian modelling approach. It is intuitive to expect that if an appropriate model that is able to exploit correlation asymmetries is used, the FSO method as applied to utility functions with higher-moment preferences should be improved due to the resulting reduction in the degree of estimation error.

Specifically, we apply a mathematical model that allows for asymmetries in the dependence structure, volatility and skewness to generate estimates of expected returns for input into FSO. Asymmetries in the dependence structure are captured using the Clayton CVC. A GARCH-GJR (Glosten *et al.*, 1993) and the skewed Student t (Skew-T) are used to capture asymmetric volatility, skewness and kurtosis within the marginals. Our application of FSO involves a range of S-curve, bilinear and kinked power utility functions. For robustness, we use a variety of parameters in our application of each utility function over a range of data sets consisting US industry and international country indices. A set of risk-adjusted return metrics are applied to provide economic intuition and indices of acceptability (Cherny and Madan, 2009) to compare the performance between using our forecasting model compared to historical sampling windows to evaluate the reduction in estimation error. The indices of acceptability are designed as ranking performance measures for evaluating the degree of consensus surrounding investment performance in the presence of non-normal and nonlinear returns distributions. Thus, they are useful for investors with higher-moment preferences and the evaluation of hedge fund performance (Eberlein and Madan, 2009).

Our results show that the Clayton CVC is able to generate a US Asymmetric Dependence Index that increases sharply during notable crisis periods in the US market. This indicates that during crisis periods, asymmetric correlations increase significantly. Furthermore, we find that compared to historical returns, accounting for returns asymmetries in the forecasting process is able to produce greater risk-adjusted returns and indices of acceptability for investors with higher-moment preferences. Management of correlation asymmetries reduces the degree of error in the estimation of expected returns therefore successfully improving FSO. We find that kinked power utility investors (i.e. conservative investors) have Sharpe ratios ranging from 1.07 to 8.27 with historical returns samples, and this improves to 8.65 to 11.64 with asymmetric returns estimates. For S-curve utility investors (i.e. aggressive investors), the Sharpe ratios range from −4.11 to 1.37 with historical returns samples, and this improves to 9.57 to 14.91 with asymmetric returns estimates. The indices of acceptability as measured by AIMAXMIN range from 1.98 to 3.35 (2.33 to 5.10) for kinked power (S-curve) utility investors. Thus, we find that investors who exhibit the least caution in their investment approach have the greatest economic benefits as indicated by risk-adjusted returns when asymmetric returns estimates are used. Similarly, investors who are conservative also exhibit higher scores across the indices of acceptability.

The novelty of our contribution is threefold. First, we use the Clayton CVC to generate an Asymmetric Dependence Index for the US market that shows that asymmetric correlations increase during notable crisis periods. Second, we reduce estimation error by applying the FSO methodology incorporating higher-moment investor preferences, with returns forecasts that account for asymmetries within the dependence structure and marginals using the Clayton CVC. Vine copulas are found to be useful in forecasting value-at-risk (VaR) and minimising conditional value-at-risk (CVaR) (Low *et al.*, 2013). We extend this literature to evaluate the benefits of vine copulas in modelling asymmetric correlations in constructing portfolios for investors who require higher-moment risk premiums. Successfully managing asymmetric correlations is valuable as Wang and Bidarkota (2010) show that over the long run, the presence of fat tails in financial and macroeconomic time series results in risk-averse agents demanding a higher equity premium to compensate the increased frequency of extreme events. Third, previous studies on FSO have used historical data in out-of-sample, single-period studies of 10 years or less with no short sales (Adler and Kritzman, 2007; Hagströmer *et al.*, 2008; Hagströmer and Binner, 2009). We build upon this literature by implementing a more rigorous study of FSO, for a variety of utility functions on several data sets over a long-term investor horizon in a tactical asset allocation exercise that allows for portfolio rebalancing and short sales.

The paper is organised as follows. Section 2 describes our data set that consists of portfolios of US and international country indices. Section 3 outlines the Clayton copula and the vine copula models. The FSO methodology and the utility functions applied are detailed in Section 4. Section 5 describes the research method used to examine the persistence of asymmetric correlations using the Clayton CVC in an out-of-sample portfolio optimisation investigation. In Section 6, we present the empirical results of our study regarding the out-of-sample performance of applying the Clayton CVC in higher-moment portfolio optimisation applications with FSO. We summarise and conclude our work in Section 7.

2. Data

Our investigation is performed on US industry equity indices. We use arithmetic returns in excess of the risk-free rate[1] and US industry portfolios.[2] The full time series sample of all constituent assets of the US portfolios rejects the Jarque–Bera test of normality at the 1 percent level.[3] We use rolling-sampling windows of historical returns of 10 years for direct input into the optimisation model or to parameterise our models for estimating expected returns. This results in an out-of-sample period of 26 years for the US industry indices data set. Details of our sample period and the data source are shown in Table 1.

By allocating wealth across portfolios of indices rather than individual stocks, these diversified portfolios are readily investable as index futures. As a result, they exhibit lower idiosyncratic risk, higher liquidity, minimal transaction costs (Balduzzi and Lynch, 1999), the absence of short-sales constraints (Chan and Lakonishok, 1993) and lower adverse selection costs (Berkman *et al.*, 2005). As excessive downside correlations have been reported in US equities (Ang and Chen, 2002) and international country indices (Longin and Solnik, 1995, 2001),[4] an application of asymmetric copulas should produce a superior fit for these data sets (Low *et al.*, 2013).

Table 1
List of data sets

Name	Source	Full sample N	Full sample time period	Out of sample time period
US industries	Ken French's website	17	07/1963–12/2010	07/1973–12/2010

This table shows the list of data sets used in our study and their sources. *N* denotes the total number of risky assets within the portfolio.

[1] We use the 1-month Treasury bill ratefrom Ibbotson Associates as provided on Ken French's website.

[2] The 17 US industries consist of Food, Mines, Oil, Clothing,Durables, Chemicals, Consumables, Construction, Steel, Fabricated Products,Machinery, Cars, Transportation, Utilities, Retail, Finance, and Other. As a robustness check, we have applied our study on a portfolio of 12 US industries and can provide these results upon request.

[3] For reasons of brevity, we do not report the descriptive statistics of our US industry portfolio. These can be provided upon request.

[4] We also apply our investigation on a portfolio of international country indices. The results can be found in Appendix VII.

3. Clayton canonical vine copula model

A copula[5] allows the flexible modelling of the dependence structure and marginals in a multivariate probability model. To understand the concept of copulas intuitively, modelling a portfolio of assets requires modelling each asset individually and the interactions between each asset. Modelling these interactions is performed with the dependence structure, and modelling each individual asset's characteristics is performed in the marginals.

We apply the Clayton CVC that allows for lower (left) tail dependence that is able to capture the increased downside correlations that often occur during a bear market regime. Asymmetries within the marginals are captured using the GARCH-GJR (Glosten *et al.*, 1993) for volatility and Skew-T (Hansen, 1994) model for skewness and kurtosis within the residuals.[6]

In Section 3.1, we show the existence of correlation asymmetries on the US equities market and how the Clayton copula models this empirical artefact in the returns distributions more accurately than Pearson's correlation and the VCV matrix. Section 3.2 describes the hierarchal structure of the vine copula that forms a joint distribution of all assets within the portfolio.

3.1. Asymmetric correlations and the Clayton copula

As a range of financial literatures report (Longin and Solnik, 1995, 2001; Ang and Chen, 2002) that negative returns are more prone to dependency than positive returns within the equities asset class, we require a model that is able to capture left (lower) tail dependence. Patton (2004) shows that the Clayton copula is one of the simplest copulas that is able to model lower tail dependence in financial time series, and Hong *et al.* (2007) use the Clayton copula in the simulation approach to analyse the performance of a model-free statistical test for identifying asymmetric correlations. The density of the bivariate Clayton copula is shown in Equation (1).

$$C_\alpha(u_1, u_2) = (1+\alpha)(u_1 \cdot u_2)^{-1-\alpha} \times \left[u_1^{-\alpha} + u_2^{-\alpha} - 1\right]^{-1/\alpha-2}, \tag{1}$$

where α is the parameter controlling the degree of lower tail dependence. Perfect dependence is obtained when $\alpha \to \infty$, and $\alpha \to 0$ implies independence.

Statistical tests of asymmetric dependence named 'exceedance correlations' are presented by Longin and Solnik (1995, 2001) and Ang and Chen (2002) who use them to report left (lower) tail dependence on international and US equities markets, respectively. The intuition of 'exceedance correlations' is that beyond a certain threshold, the correlation behaviour across assets changes. Similar

[5] A more detailed derivation of the CVC can be found in the Appendix I.

[6] Details of themarginal models are provided in the Appendix II.

ideas are applied in extreme value theory (EVT) where the behaviour of returns in the tails of a distribution can be different from returns within the main body of the distribution and therefore require different statistical assumptions and modelling techniques (Embrechts *et al*., 1999, 2013). Patton (2004) and Chua *et al*. (2009) apply exceedance correlation tests to indicate existence of asymmetric correlations on their data sets, and propose different techniques to manage them in the context of portfolio optimisation. Patton (2004) applies copulas to forecast asymmetric dependence, and Chua *et al*. (2009) apply FSO with a kinked utility function. Hong *et al*. (2007) provide a model-free test of asymmetric correlations and show that incorporating asymmetries in investment decisions is of economic importance for investors with disappointment aversion preferences.

The existing literature (Ang and Chen, 2002; Patton, 2004; Hong *et al*., 2007; Chua *et al*., 2009) has applied statistical tests to prove the existence of asymmetric correlations in our data set of US equities. Thus, we focus on providing intuitive graphical analysis to explain the phenomenon of asymmetric correlations based on the premise of 'exceedance correlations' where correlation behaviour changes beyond an 'exceedance threshold' in Figures 1 and 2. If a portfolio of assets exhibits elliptical dependence, the behaviour of returns beyond the exceedance thresholds is such that there should be an equal

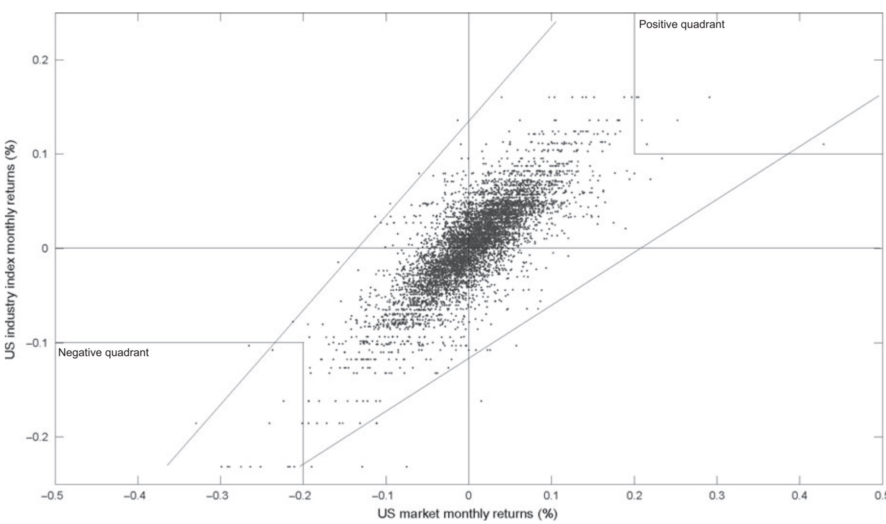

Figure 1 Asymmetric dependence (correlation) in US equities market. This figure shows a scatter plot of the returns of US industries vs US market returns from 1962 to 2010. For the US market monthly returns, it shows thresholds of −0.2 (−20 percent) that denotes the negative quadrant, and +0.2 (20 percent) as the positive quadrant. There are a higher number of points in the negative quadrant than in the positive quadrant. Therefore, there are greater correlations in US market downturns as opposed to market upturns, that is indicative of left (lower) tail dependence.

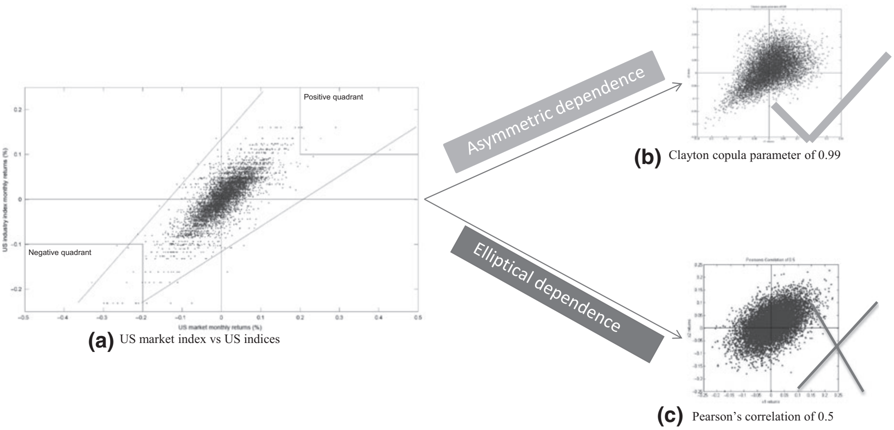

Figure 2 US asymmetric dependence index. This figure shows the evolution of asymmetric dependence across 48 industry indices that comprise the US equities market. By summating the Clayton copula parameter across the entire vine copula, we obtain an aggregate indicator of asymmetric dependence (correlation) across the US market. It is observed that asymmetric dependence increases dramatically during notable crisis events in the US market.

number of points in the positive and negative quadrants with equal dispersion. If a portfolio of assets exhibits left (lower) tail dependence, there will be a greater number of points in the negative quadrant in a denser, concentrated region compared to the positive quadrant (Embrechts *et al.*, 2002).

Figure 1 is a scatter plot of monthly returns from 1962 to 2010 of 12 US industry indices that together comprise the US equities market versus the US S&P 500 index (market index). We apply exceedance thresholds of +0.2 (+20 percent) and −0.2 (−20 percent) for the US market index. Returns that are beyond this threshold exhibit a different correlation structure than the main body that lies within these thresholds. We denote the returns beyond these thresholds as positive and negative quadrants, respectively. We can see that in the negative quadrants, there are a greater number of points compared to the positive quadrant. Thus, this indicates the existence of left tail dependence in US equities. The lines are used to indicate that in the positive quadrants, there is a greater dispersion of points compared to the negative quadrant where it is more concentrated. Figure 2 shows that the long-term behaviour of US industry indices comprising the entire US market exhibits asymmetric correlations that are better modelled using a Clayton copula in Figure 1b that allows for left tail dependence. The VCV that uses Pearson's correlation, as shown in Figure 1c, is inferior as it only measures elliptical dependence.

Figure 3 is a graphical representation of correlations between a simulated pair of assets (x_1 and x_2) based upon the Clayton copula and Pearson's correlation. As the Clayton copula parameter increases, the degree of

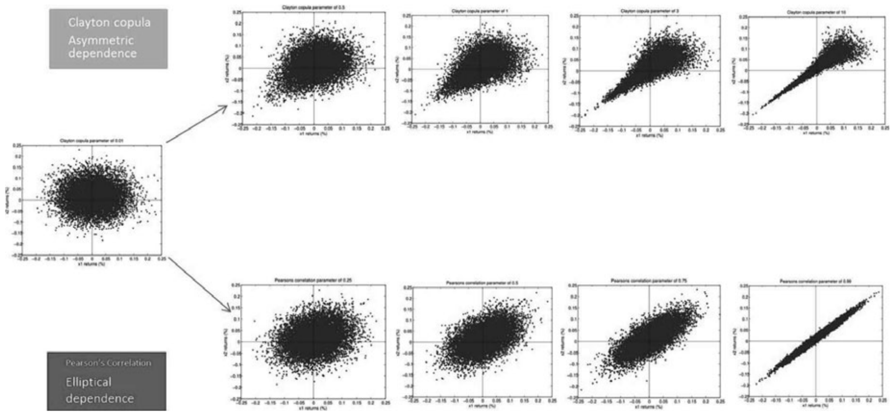

Figure 3 Representation of asymmetric and elliptical dependence across two assets as given by the Clayton copula parameter and Pearson's correlation, respectively. This figure shows the evolution of asymmetric and elliptical dependence as given by the Clayton copula and Pearson's correlation when the respective parameters increase. As the Clayton copula parameter increases from to 10, asymmetric dependence (correlation) is observed where there is an increase in lower tail dependence, but not upper tail dependence. As the Pearson's correlation parameter increases, the dependence is symmetric and the correlations are equal in the positive and negative quadrants.

asymmetric correlations increases. Intuitively, in the negative returns quadrant, we observe increased correlations, whereas in the positive returns quadrant, the correlations between both assets remain dispersed. Alternatively, as the Pearson's correlation increases, we can see that both left and right tail correlations increase equally. Thus, the Clayton copula is able to capture asymmetric correlations that are observed to occur in the US and international equities markets, whereas the traditional VCV that applies Pearson's correlation as a measure of dependence is unable to capture the effect of such asymmetric correlations.

3.2. Canonical vine copula hierarchal structure

The CVC applies a hierarchal vine to link copula pairs together in a multivariate probability model.[7] If key assets that govern the interactions in the investment portfolio can be identified during the modelling process, it is possible to locate these variables towards the 'root' of the canonical vine. Thus, we are able to build the canonical vine by ordering assets closer to the root of the structure by their degree of correlation with other assets within the portfolio.

[7] For details regarding other vinestructures such as the D-vine, see Aas *et al.* (2009).

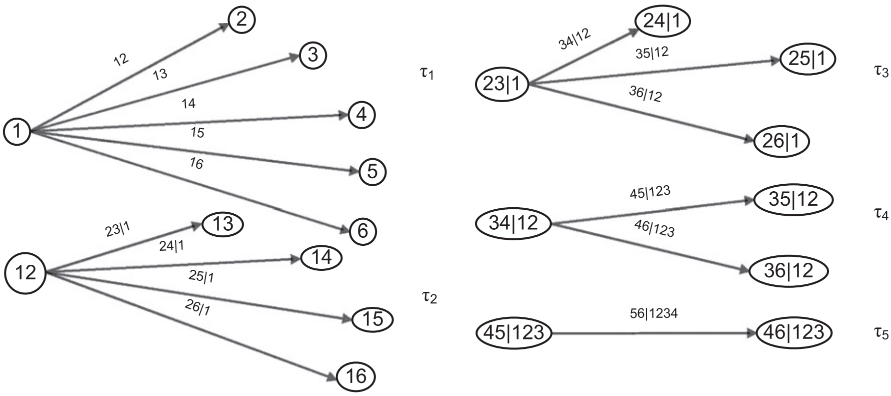

Figure 4 Canonical vine copula hierarchal structure for six asset portfolios. This figure shows the canonical vine copula structure for a hypothetical portfolio of six assets. The dependence structure is comprised of a hierarchal set of 5 trees, 15 edges and 20 nodes. As each edge corresponds to a bivariate copula density, there are a total of 15 bivariate copulas. Tree τ_1 consists of five bivariate copulas of asset 1 and each asset from 2 to 6. Tree τ_2 consists of four bivariate copulas of asset 2 with each asset 3 to 6, based conditionally upon asset 1. Tree τ_3 shows the modelling of three bivariate copulas of asset 3 with each asset 4, 5 and 6, based conditionally upon assets 1 and 2. Tree τ_4 shows the modelling of two bivariate copulas of asset 4 with each asset 5 and 6, based conditionally upon assets 1, 2 and 3. Tree τ_6 consists of a bivariate copula between assets 5 and 6, based conditionally upon assets 1 to 4.

Figure 4 shows a CVC structure for a hypothetical portfolio of N = 6 assets, with a total of five trees, τ_k, where k = 1, … 5. Each tree, τ_k, contains $7 - k$ nodes and $6 - k$ edges (edges are the connections between two nodes). The label of each edge is the subscript of the copula pair density. For example, the edge label 46|123 corresponds to the copula density 46|123. Each node in tree τ_k determines the labels of the edges in tree τ_{k+1}. When two edges in τ_j both share a common node, they become nodes in τ_{j+1} and are joined by an edge. For example, in τ_1, edges 12 and 13 share a common node in asset 1. Therefore in τ_2, they both become nodes and share the edge 23|1. Thus, the entire model's density is decomposed to N $(N - 1)/2$ edges and each asset's marginal densities. For example, the asymmetric dependence structure characterised by a Clayton CVC model for a portfolio of six assets requires the parameterisation of N $(N - 1)/2$ variables.

As we consider asset 1 to be a key asset that is related to all other assets within the portfolio, we place it at the 'root' of the CVC structure in tree τ_1. We use five bivariate copulas to capture the dependence structure between asset 1 and each asset from 2 to 6. Asset 2 is considered to be the second most important asset governing interactions within the portfolio. Thus, τ_2 captures the relationships between asset 2 and assets 3 to 6, conditionally upon asset 1. For example, 23|1 denotes the bivariate copula between assets

2 and 3, based conditionally upon asset 1[8]. Thus, we order our assets accordingly such that the most important asset that is believed to govern interactions within the portfolio belongs at the 'root' of the CVC. Other assets placed close to the 'root' of the CVC are done so with decreasing importance within the portfolio.

In our implementation, we follow Low *et al.* (2013) in designing the canonical vine structure by placing assets that have the highest degree of linear correlation with all the other assets in the sample window at the root of the structure. Specifically, this is performed by calculating the average of the Pearson's correlation matrix between all assets during the sample window as shown in Equation 2.

$$\bar{\Theta}_y = \frac{1}{N}\sum_{x=1}^{N}\theta_{xy}, \tag{2}$$

where θ_{xy} is an $N \times N$ matrix of the Pearson's correlation parameter of the monthly returns between each pair of assets x and y that are both part of our portfolio of N assets. Θ_y is a $N \times 1$ matrix where each element is the sum of the Pearson's correlation parameter of y with all other assets x. Θ_y is sorted in descending order to facilitate the process of placing assets with the highest value of correlations closest to the root of the canonical vine structure. For example, the largest value in Θ_y has the highest absolute linear correlation with all other assets within the portfolio during the sample window and is placed at the root of the hierarchal structure of the canonical vine.

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[8] By construct, $C_{23|1}$ is different from C_{23}. The vine copula approach uses conditional pair-copulas that are assumed to depend on conditioning variables indirectly through the conditional margins. Hobæk Haff *et al.* (2010) show that the construction of the vine based on pair-copulas is a good approximation, even when the simplifying assumption is far from being fulfilled by the actual model. Acar *et al.* (2012) discuss that although the simplifying assumptions can be misleading, they agree that it is the only methodology that currently exists that offers statisticians great flexibility in modelling multivariate dependence for high dimensions. Therefore, the vine copula continues to be heavily applied in applied financial research. For studies where vine copulas are used to analyse statistical and time series properties of financial returns, see Heinen and Valdesogo Robles (2009), Nikoloulopoulos *et al.* (2012), Dissmann *et al.* (2013) and Brechmann *et al.* (2012). The vine copula has been used extensively in a variety of practical applications in finance such as modelling asymmetric dependence for investors who seek to minimise conditional value-at-risk (CVaR) (Low *et al.*, 2013), forecasting VaR (value-at-risk) (Weiβ and Supper, 2013; Zhang *et al.*, 2014), risk management (Brechmann and Czado, 2013; Brechmann *et al.*, 2014) and exchange rates (Czado *et al.*, 2012).

4. Full-scale optimisation

In practice, portfolio optimisation invokes several simplifications such as ignoring higher-order moments, using parameter estimates rather than predictive returns, maximising approximations of the expected utility, and estimation error (Adler and Kritzman, 2007; Hagströmer *et al*., 2008; Harvey *et al*., 2010). Such simplifications can lead to inferior investment decisions as behavioural finance studies show that managing higher moments in investment portfolios is of increasing importance to investors (Kahneman and Tversky, 1979; Thaler *et al*., 1997; Odean, 1998) and is crucial to minimising exposure to downside risk (Chua *et al*., 2009; Low *et al*., 2013).

We apply FSO[9] as a potential solution to the first three simplifications and address the issue of estimation error in expected returns by applying a CVC model that incorporates asymmetries in the dependence structure and marginals. We seek to demonstrate the improvement in performance outcomes for 'sophisticated' investors who are able to exploit the persistence of correlation asymmetries in equities returns to produce superior estimates of expected returns versus 'naïve' investors who use unadjusted historical sampling windows. Our study applies FSO to these different estimates of expected returns on portfolio choice for investors with varying risk preferences to further analyse the outcomes of risk-seeking versus conservative investment behaviour when asymmetric estimates are applied.

We explore a range of alternative utility functions that exhibit preferences for higher moments (e.g. kinked power) and provide their functional forms in Table 2. Graphical representations of the kinked power, bilinear and S-curve utility functions for a range of parameters are shown in Figures 5, 6 and 7, respectively.

Our study investigates a total of 30 utility functions[10] for each investment portfolio. However for brevity, we report the results for 27 portfolios as the results for the kinked power utility function where $\gamma = 1$ and $\lambda = 1$ are quantitatively similar to those for the bilinear utility function when $p = 1$. Table 3 provides details of the parameters used for each utility function, the number of utility function specifications under investigation and the parameters applied in a range of extant FSO studies.[11] As each utility

[9] Details on the implementation of FSO are given in Appendix III. We use the acronym FSO as this is the label applied for the approach in the literature (Adler and Kritzman, 2007; Hagströmer *et al*., 2008; Chua *et al*., 2009; Hagströmer and Binner, 2009; Kritzman, 2011).

[10] A detailed description of the utility functions explored is given in Appendix IV.

[11] We report a range of parameters applied in FSO studies by Cremers *et al*. (2003), Adler and Kritzman (2007), Hagströmer *et al*. (2008), Hagströmer and Binner (2009) and Chua *et al*. (2009) for each of the utility functions investigated in our study.

Table 2
Utility functions investigated

Utility type	Functional form	Conditions
Kinked Power	$\ln(1+x)$	$x \geq \theta, \gamma = 1$
	$\ln(1+\theta-\lambda[\theta-x])$	$x < \theta, \gamma = 1$
	$[(1+x)^{1-\gamma}-1](1-\gamma)^{-1}$	$x \geq \theta, \gamma > 0$
	$[(1+\theta-\lambda[\theta-x])^{1-\gamma}-1](1-\gamma)^{-1}$	$x < \theta, \gamma > 0$
Bilinear	$\ln(1+x)$	$x \geq \theta$
	$P(x-\theta)+\ln(1+\theta)$	$x < \theta$
S-curve	$-A(\theta-x)^{\tau_1}$	$x \leq \theta$
	$+B(x-\theta)^{\tau_2}$	$x > \theta$

This table shows the list of utility functions. Portfolio returns are denoted by x. The critical level of returns that is the inflection point for S-curve utility and the kink point for bilinear and kinked power utility is θ. For bilinear utility, P is the penalty level on sub-kink returns. In kinked power utility functions, γ is the degree of relative risk aversion (RRA), and λ is the degree of loss aversion. For S-curve utility functions, τ_1 (τ_2) and A (B) respectively determine the shape and the magnitude of the downside (upside) region of the function.

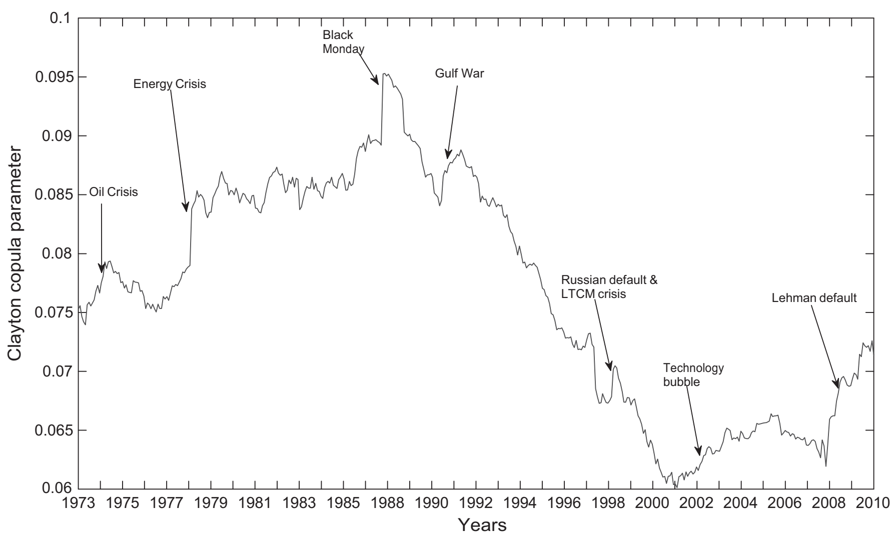

Figure 5 Kinked power utility – illustrative cases. This figure illustrates kinked power utility functions with varying values for the degree of relative risk aversion (γ) and loss aversion (λ). The kink point (θ) is set at 0 percent.

function requires different sets of parameters, we cover a range of reasonable values to allow for a meaningful comparison across all utility functions investigated.

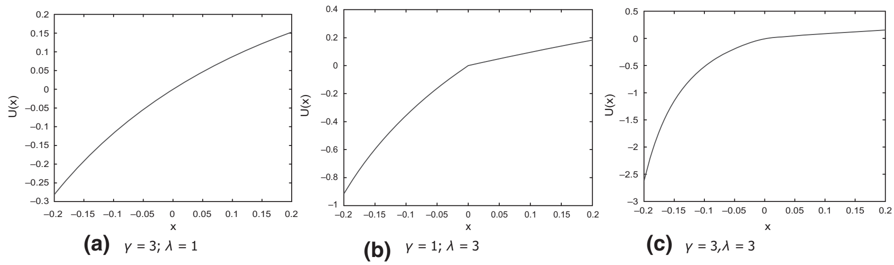

Figure 6 Bilinear utility – illustrative cases. This figure illustrates bilinear utility functions with varying values for the penalty (P) applied to the subkink returns. The kink point (θ) is set at 0 percent.

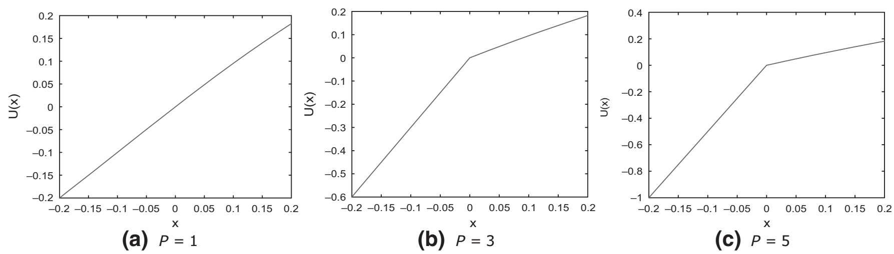

Figure 7 Modelling asymmetric dependence (correlation) with copulas. This figure shows (a) US market scatter plot, (b) scatter plot produced by a Clayton copula parameter of 0.99 and (c) scatter plot produced with a Gaussian copula with correlation parameter of 0.5. The Clayton copula exhibits a larger number of points in the negative quadrant than in the positive quadrant, thus indicating its ability to model left (lower) tail dependence. Elliptical dependence has an equal number of points in both the positive and negative quadrants and is unable to allow for asymmetric dependence. Thus, the Clayton copula leads to an improved model of the US market due to its ability to model asymmetric dependence.

5. Empirical testing procedure

Our research method simulates a typical scenario faced by portfolio managers in a tactical asset allocation strategy where the portfolio is dynamically rebalanced each month to maximise a selected utility function as new information arrives from each asset (Brands *et al.*, 2006; Hatherley and Alcock, 2007; Chiang and Zhou, 2009). Specifically, our approach estimates expected returns only on the basis of information available at the time of portfolio construction; thus, every estimate is out-of-sample. We use rolling-sampling windows where if the entire data set of asset returns consists of T months, the out-of-sample period consists of $T - W$, where W is the size of the

Table 3
Utility function parameters

Utility function	Number of utility function specifications	Parameter values	Parameter values of prior FSO studies
Kinked power	12	$\theta = 0\%\ -2\%,\ -5\%$ $\gamma = 1,\ 3;\ \lambda = 1,\ 3$	$-1.5\% \leq \theta \leq 0.5\%$ $1 \leq \gamma \leq 6;\ 1 \leq \lambda \leq 3$
Bilinear	9	$\theta = 0\%,\ -2\%,\ -5\%$ $P = 1,\ 3,\ 5$	$-5\% \leq \theta \leq 0.5\%$ $1 \leq p \leq 10$
S-curve	9	$\theta = 0\%,\ -2\%,\ -5\%$ $B/A = 1,\ 2,\ 3;\ \tau_1 = \tau_2 = 0.5$	$-5\% \leq \theta \leq 0\%$ $1 \leq B/A \leq 3;\ 0.1 \leq \tau_1,\ \tau_2 \leq 0.5$

This table shows the parameters applied to the utility functions that we optimize for in our study. As a result of the selected combination of parameters, we show the total number of combinations of utility functions investigated. We also report the range of parameter values applied in prior FSO studies.

sample window of 120 months.[12] During each month t, starting from $t = W + 1$, data within the previous W months are used to generate expected returns. To evaluate the degree of estimation error, we generate estimates of expected returns in two ways. First, as a benchmark, expected returns are simply unadjusted historical rolling sample windows that are input into FSO. Second, to reduce estimation error, the same rolling sample windows are used to parameterise the model estimates of expected returns. Once the model is parameterised, we use Monte Carlo simulations to generate 10,000 returns for each asset within the portfolio. The sample of simulated returns is then used as an input into the optimisation models. To summarise, as our portfolio consists of 17 assets, our model simulated 170,000 returns per month. Our out-of-sample period consists of 450 months; thus, the total size of our simulation data set has a total of 76.5 million returns observations. We then compare the portfolio performance of generating expected returns between using unadjusted historical samples or model estimates of expected returns incorporating asymmetric dependence across a range of utility functions.

6. Results

First, we show that the Clayton CVC is able to capture changes in asymmetric dependence in the US market and this can be represented

[12] Our approach of using a 10 year rolling window follows the empirical research method applied in the portfolio optimisation literature (DeMiguel *et al.*, 2009; Fletcher, 2011; Tu and Zhou, 2011; Low *et al.*, 2013, 2016). Volatility clustering effects in the marginals are captured through the GARCH-GJR model and skewness and kurtosis by our application of the Skew-T model. Our results are robust to the use of 20 year (240 month) rolling windows and can provide these results upon request.

graphically in terms of a US Asymmetric dependence index. Second, we measure out-of-sample performance by reporting a set of risk-adjusted return (economic measures) metrics and the indices of acceptability (consensus measures) for our data sets.[13] For descriptive ease, we denote alternative utility functions as follows. Kinked utility specifications are generally indicated by "K", and subscripts identify γ and λ, respectively. For example, "$K_{1,3}$" denotes a kinked utility function with $\gamma = 1$ and $\lambda = 3$. Bilinear utility functions are generally indicated by "B", and the subscript identifies the order of P. For example, "B_3" denotes a bilinear utility function where $P = 3$. S-curve utility specifications are generally indicated by 'S', and the subscript identifies the ratio of B/A. For example, "S_3" denotes an S-curve utility function where $B/A = 3$. When the kink/inflection point is such that $\theta = -2$ percent, this will be denoted as '$\theta_{-2\text{ percent}}$'.

6.1. US asymmetric dependence index

Figure 2 displays a US Asymmetric Dependence Index. The Clayton copula parameter is extracted from each pair as defined within the hierarchal structure of the CVC within the portfolio of assets and aggregated to obtain a measure of asymmetric correlations across the entire portfolio. We apply the monthly returns from US industry indices that comprise the US market portfolio, and we use 10 year rolling-sampling windows to parameterise the Clayton CVC. Thus, the US Asymmetric Dependence Index is a 1 month-ahead forecast based on 10 year monthly returns data that are used to parameterise the Clayton CVC. During the period of 1963–2010, the degree of asymmetric correlations has decreased over time. However, during notable crisis periods in the US markets, sharp increases in the degree of asymmetric correlations across US industries are observed.

As can be seen from the US Asymmetric dependence index in Figure 2, the Clayton CVC can be applied in a similar manner; however, our study focuses on using the Clayton CVC to generate forecasts of returns to model the persistence of asymmetric correlations that are vital in reducing risk exposures in investment portfolios. Adler and Kritzman (2007) and Hagströmer *et al.* (2008) find that the successful out-of-sample performance of FSO is dependent upon the persistence of the properties of the sample return distributions used to calculate the optimal asset allocations. Thus, we show that the Clayton CVC is able to capture the persistence in asymmetric correlations that improves the out-of-sample performance outcomes in the FSO methodology (Figure 8).

[13] More details regarding our choices of economic and consensus measures can be found in Appendix V. Selection of measurements for portfolio performance is important for rating different funds and portfolio strategies (Gerrans, 2006).

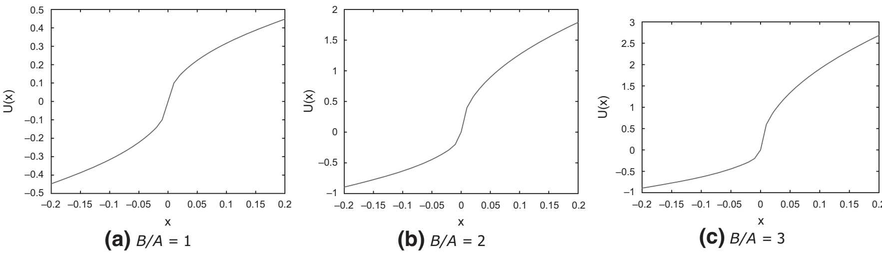

Figure 8 S-curve utility – illustrative cases. This figure illustrates S-curve utility functions with varying values for B/A. The inflection point (θ) is set at 0 percent. The shape and magnitude of the downside (upside) region on the left of the inflection point are given by τ_1 (τ_2) and A (B), respectively. Both τ_1 and τ_2 are set to 0.5.

6.2. *Risk-adjusted return metrics*

We apply a range of popular risk-adjusted return metrics to provide economic intuition regarding the benefits of using asymmetric estimates for different utility functions. We report the Sharpe ratio as it is a widely used measure of portfolio performance in industry and academia, as well as other popular downside risk-adjusted measures (also known as gain–loss ratios) such as the Omega ratio (Keating and Shadwick, 2002) and the Sortino ratio (Sortino and Van Der Meer, 1991). In addition, to evaluate the performance relative to extreme downside exposure, we calculate the ratio of the average returns of the portfolio strategy relative to the 1 percent level of CVaR (denoted as Mean/CVaR), that is a coherent risk-adjusted return on capital.

6.2.1. *US industry setting*

Table 4 shows risk-adjusted returns for the portfolio of 17 US industry indices. The performance for the different utility functions is much poorer when historical returns are applied compared to asymmetric estimates. The difference is greatest for the S-curve utility functions that produce negative returns when historical samples are used. Generally, all utility functions are significantly improved across all risk-adjusted metrics when asymmetric estimates are used. We find that kinked power utility functions exhibit the best performance with historical returns. When asymmetric returns are applied, bilinear and S-curve utility functions exhibit the best performance and improvements, respectively.

In Panel A (kinked power utility), the most conservative strategies with the highest loss aversion parameters, $K_{3,3}$ and $K_{1,3}$ produce the highest risk-adjusted returns for all values of θ. For $K_{3,3}$ ($K_{1,3}$) they produce Sharpe ratios ranging from 5.85 to 8.27 (5.86 to 6.21). They also exhibit the least improvement when asymmetric estimates are used, where $K_{3,3}$ ($K_{1,3}$) produces Sharpe ratios ranging from 8.65 to 10.61 (10.59 to 10.74). On the other hand,

Table 4
Risk-adjusted return metrics for FSO utility functions – US industry indices analysis

Utility parameters		Sharpe ratio		Omega ratio		Sortino ratio		$\frac{\text{Mean}}{\text{CVaR}}$	
θ (%)	Risk parameters	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.
Panel A: Kinked power utility									
0	$K_{1,3}$	6.04	10.59	1.18	1.33	8.56	15.30	1.82	2.92
	$K_{3,3}$	7.11	8.65	1.21	1.27	10.29	13.21	2.32	2.90
	$K_{3,1}$	1.07	10.98	1.03	1.36	1.50	16.20	0.30	3.00
−2	$K_{1,3}$	5.86	10.64	1.17	1.33	8.11	15.45	1.57	2.94
	$K_{3,3}$	8.27	10.61	1.24	1.34	11.94	15.66	2.64	3.08
	$K_{3,1}$	1.07	11.64	1.03	1.38	1.50	17.26	0.30	3.17
−5	$K_{1,3}$	6.21	10.74	1.19	1.34	8.88	16.04	1.78	2.84
	$K_{3,3}$	5.85	9.55	1.17	1.29	8.32	13.90	1.75	2.69
	$K_{3,1}$	1.07	11.44	1.03	1.37	1.50	16.95	0.30	3.14
Panel B: bilinear utility									
0	B_1	−4.03	15.75	0.89	1.53	−5.36	25.10	−1.04	5.47
	B_3	3.95	12.83	1.12	1.41	5.43	19.00	1.08	3.56
	B_5	4.96	12.29	1.14	1.39	7.13	17.98	1.61	3.42
−2	B_1	−4.02	15.72	0.89	1.53	−5.35	25.04	−1.04	5.46
	B_3	5.62	12.37	1.17	1.40	7.76	18.32	1.45	3.30
	B_5	8.11	11.79	1.24	1.37	11.93	17.24	2.79	3.34
−5	B_1	−4.08	15.68	0.89	1.53	−5.42	24.96	−1.06	5.44
	B_3	5.27	12.09	1.16	1.40	7.46	17.93	1.44	3.03
	B_5	5.75	11.35	1.17	1.35	8.06	16.58	1.70	3.04
Panel C: S-curve utility									
0	S_3	−3.13	9.57	0.92	1.30	−4.22	14.31	−0.94	2.86
	S_2	−3.50	12.44	0.91	1.41	−4.69	19.19	−0.98	3.64
	S_1	−3.45	14.91	0.91	1.51	−4.69	23.81	−1.03	4.99
−2	S_3	−3.67	13.51	0.91	1.46	−4.97	20.74	−1.05	3.94
	S_2	−3.84	13.54	0.90	1.46	−5.12	20.83	−1.06	4.04
	S_1	−1.76	12.03	0.95	1.43	−2.36	18.67	−0.49	3.43
−5	S_3	−0.92	14.00	0.97	1.50	−1.26	22.20	−0.26	4.18
	S_2	−4.11	11.30	0.89	1.40	−5.36	17.92	−1.06	3.77
	S_1	1.37	13.74	1.04	1.47	1.90	21.43	0.40	3.96

This table shows the Sharpe, Omega, Sortino, and Mean/CVaR ratios for investors with kinked power (Panel A), bilinear (Panel B) and S-curve (Panel C) utility functions for a portfolio of 17 US industry indices when historical samples or asymmetric estimates of expected returns are applied. The kink/inflection point of the utility function is denoted by θ.

$K_{3,1}$ that exhibits a Sharpe ratio range of around 1.07 is improved to 10.98 to 11.64. Therefore, as asymmetric estimates impose a degree of conservativeness in the portfolio decision process, $K_{3,1}$, that is the most risk-seeking of all the kinked power utility functions explored, benefits the most.

In Panel B (bilinear utility), the two best performing bilinear utility functions are B_3 and B_5, and when historical returns are used, the resulting Sortino ratio ranges from 5.43 to 7.76, and 7.13 to 11.93, respectively. When asymmetric estimates are used, the Sortino ratio of B_3 improves to a (B_5) range of 17.93 to 19.00 (16.58 to 17.98), thus exhibiting substantial improvement. Although B_1 is the poorest performing utility function and generates negative returns with historical returns samples (Sortino ratio between −5.35 and −5.42), it exhibits the largest gains and outperforms the two more loss-averse counterparts when asymmetric returns are accounted for (Sortino ratio between 24.96 and 25.10). Thus, the weaker the degree of loss aversion, the greater the benefits to be gained from using asymmetric returns estimates.

In Panel C (S-curve utility), the best (worst) performing S-curve utility functions when historical returns are applied is S_1 (S_2). However, Panel C differs from the other panels as, regardless of the value of θ, the degree of improvement with the application of asymmetric returns across all S-curve utility functions is very similar. All of them experience a large magnitude of improvement such that from being the poorest performing set of utility functions with historical returns (Sharpe ratios of −4.11 to 1.37), they outperform the kinked power utility functions when asymmetric returns estimates are used (Sharpe ratios of 9.57 to 14.91). Therefore, we still observe that the most (least) conservative strategies perform much better when historical returns (asymmetric estimates) are applied.

6.3. Indices of acceptability performance outcomes

Cherny and Madan (2009) state that a measure of trading performance should satisfy the following axioms: (i) quasi-concavity; (ii) monotonicity; (iii) scale invariance; (iv) Fatou property; (v) law invariance; (vi) consistency with second-order stochastic dominance; (vii) arbitrage consistency; and (viii) expectation consistency. Several popular risk-adjusted portfolio performance measures such as the Sharpe ratio, gain–loss ratios (e.g. Sortino ratio) and coherent risk-adjusted return on capital (e.g. Mean/CVaR) are able to satisfy some but not all of the above axioms. Thus, Cherny and Madan (2009) recommend the AIMIN, AIMAX, AIMINMAX and AIMAXMIN as promising new measures for investment performance evaluation as they are able to satisfy all eight axioms, where 'AI' denotes acceptability index. Eberlein and Madan (2009) apply the indices of acceptability as a performance measure for hedge funds. In addition, indices of acceptability have been used to price and optimally hedge complex contingent claims and price corporate securities (Madan, 2010; Madan and Schoutens, 2011).

6.3.1. US industry setting

Table 5 shows a range of indices of acceptability for the portfolio of 17 US industry indices. We find that risk parameters have a greater impact than the kink/inflection points.

In Panel A (kinked power utility functions), we find that kinked utility functions for $K_{1,3}$ and $K_{3,3}$ produce greater indices of acceptability when asymmetric estimates (i.e. AIMAXMIN ranging from 2.10 to 2.63 and 1.98 to 2.35 for $K_{1,3}$ and $K_{3,3}$, respectively) are used compared to historical samples (i.e. AIMAXMIN ranging from 2.50 to 2.91 and 2.61 to 2.85 for $K_{1,3}$ and $K_{3,3}$, respectively). Both utility functions exemplify investors who are more averse to losses. The greatest enhancements are shown in $K_{3,3}$ due to a greater relative risk aversion (RRA). Improvements are visible for $K_{3,1}$ for AIMAX and AIMINMAX metrics. For cases where $\theta_{-2\text{ percent}}$ and $\theta_{-5\text{ percent}}$, we find similar results where enhancements are visible for $K_{3,3}$ for all indices and for $K_{3,1}$ in the AIMAX and AIMINMAX metrics. Across all values of θ, the least (most) conservative investor, $K_{3,1}$ ($K_{3,3}$), produces the highest (lowest) values for the indices.

In Panel B (bilinear utility functions), where $\theta_{0\text{ percent}}$ and $\theta_{-2\text{ percent}}$, we find that bilinear utilities of B_1 and B_5 are improved across all indices when asymmetric returns are used. Where $\theta_{-5\text{ percent}}$, this result holds for B_1. B_3 only demonstrates improvements for the AIMAX and AIMINMAX metrics in the $\theta_{0\text{ percent}}$ case. Therefore, the greatest (least) improvements are for B_3 (B_1). For $B_{3,}$ the AIMAXMIN metrics are 2.02 to 2.77 with historical returns and it improves to 2.58 to 3.06. For B_1, the AIMAXMIN metrics are 4.50 with historical returns and it improves to 4.66 with asymmetric estimates. Across all values of θ, the highest (lowest) values for the indices are produced by the least (most) conservative investor, B_1 (B_5). For $B_{1,}$ the AIMAXMIN produced is 4.50, whereas for B_5 it ranges from 2.01 to 2.43.

In Panel C, S-curve utility functions produce the highest indices compared to the other utility functions. For S_3 and S_2, improvements are found for $\theta_{0\text{ percent}}$ in AIMAX and AIMINMAX and in $\theta_{-2\text{ percent}}$ and $\theta_{-5\text{ percent}}$ for all indices. The use of asymmetric estimates demonstrates enhancements in S_1 for all indices where $\theta_{0\text{ percent}}$ and for AIMAX and AIMINMAX when $\theta_{-5\text{ percent}}$. S_3 (S_1) exhibits the largest (lowest) indices of acceptability. For S_3, AIMAXMIN ranges from 3.14 to 5.10 with historical samples and 3.76 to 4.91 with asymmetric estimates. For S_1, AIMAXMIN ranges from 2.33 to 3.85 with historical samples and 2.72 to 4.09 with asymmetric estimates. Thus, asymmetric returns estimates mainly benefit less conservative investors who have higher preferences for upside relative to downside gains.

6.4. Summary of overall portfolio performance

Regardless of whether historical returns or asymmetric estimates are used, S-curve utility functions produce the highest indices of acceptability and the

Table 5
Indices of acceptability results for FSO utility functions - US industry indices

Utility parameters		AIMIN		AIMAX		AIMINMAX		AIMAXMIN	
θ (%)	Risk parameters	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.
Panel A: Kinked power utility									
0	$K_{1,3}$	5.90	6.98	17.59	18.35	24.99	25.99	2.10	2.50
	$K_{3,3}$	5.59	7.20	17.29	18.15	24.59	25.66	1.98	2.61
	$K_{3,1}$	8.98	9.50	19.21	20.13	26.97	28.26	3.35	3.52
−2	$K_{1,3}$	6.03	7.29	17.48	18.77	24.81	26.57	2.14	2.61
	$K_{3,3}$	5.79	7.44	17.44	18.35	24.78	25.93	2.05	2.70
	$K_{3,1}$	8.98	9.50	19.21	20.13	26.97	28.26	3.35	3.52
−5	$K_{1,3}$	7.26	8.03	18.40	19.13	26.02	26.99	2.63	2.91
	$K_{3,3}$	6.55	7.84	17.90	19.14	25.37	27.02	2.35	2.85
	$K_{3,1}$	8.98	9.50	19.21	20.13	26.97	28.26	3.35	3.52
Panel B: Bilinear utility									
0	B_1	11.51	12.10	20.49	21.91	28.43	30.47	4.50	4.66
	B_3	6.23	7.20	17.83	18.49	25.30	26.17	2.22	2.58
	B_5	5.68	6.73	17.53	18.13	24.93	25.70	2.01	2.40
−2	B_1	11.50	12.10	20.47	21.89	28.40	30.44	4.50	4.66
	B_3	6.14	7.51	17.31	18.64	24.55	26.35	2.19	2.71
	B_5	5.71	6.95	17.26	18.45	24.53	26.14	2.02	2.48
−5	B_1	11.51	12.09	20.49	21.89	28.44	30.45	4.50	4.66
	B_3	7.62	8.38	18.59	19.26	26.25	27.14	2.77	3.06
	B_5	6.78	7.63	18.10	18.85	25.64	26.64	2.43	2.75
Panel C: S-curve utility									
0	S_3	12.76	12.47	21.17	21.69	29.21	30.06	5.10	4.91
	S_2	12.31	12.47	21.28	22.07	29.47	30.61	4.85	4.88
	S_1	10.15	10.68	19.98	20.54	27.92	28.65	3.85	4.09
−2	S_3	10.82	12.14	20.51	21.71	28.61	30.14	4.13	4.75
	S_2	10.27	11.52	19.94	21.65	27.85	30.12	3.91	4.50
	S_1	7.84	7.71	18.63	18.58	26.26	26.22	2.88	2.81
−5	S_3	8.40	9.77	18.54	19.66	26.05	27.47	3.14	3.76
	S_2	7.76	8.28	18.36	18.30	25.89	25.72	2.85	3.10
	S_1	6.50	7.55	17.34	18.85	24.56	26.65	2.33	2.72

This table shows the indices of acceptability metrics of AIMIN, AIMAX, AIMINMAX, and AIMAXMIN for investors with kinked power (Panel A), bilinear (Panel B) and S-curve (Panel C) utility functions a portfolio of 17 US industry indices when historical samples or asymmetric estimates of expected returns are applied. The kink/inflection point of the utility function is denoted by θ.

lowest risk-adjusted returns. Alternatively, kinked power utility functions produce the lowest indices of acceptability but the highest risk-adjusted returns. Bilinear utility functions fall in the middle of this spectrum as they do not allow for any changes in the utility function above the kink point, whereas kinked power utility functions allow changes in RRA. S-curve utility functions have higher risk-seeking preferences compared to kinked power utility functions. Therefore, the direction of trades taken by investors with S-curve utility seems preferable, although this may lead to lower economic outcomes. Alternatively, the direction of trades taken by more conservative individuals (i.e. kinked power) leads to economic benefits. We find that this result holds even within each category of utility functions explored where the more conservative the investor, the higher the economic gains and the lower the indices of acceptability.

Regarding the impact of using asymmetric estimates, we find that kinked power utility functions exhibit the largest and lowest improvements for the indices of acceptability and risk-adjusted returns, respectively. Alternatively, S-curve utility functions have larger enhancements for risk-adjusted returns and lower enhancements for the indices. The same observations are found within the utility functions where asymmetric returns produce larger improvements in the risk-adjusted metrics for less conservative investors. Thus, the greatest benefit of asymmetric estimates is improving the economic outcomes of investors as the impact on the indices of acceptability is much smaller in comparison.

Therefore, based on our analysis, if the investor lacks the ability to incorporate distributional asymmetries into their portfolio management process, a more conservative approach is more likely to generate more successful economic outcomes in the long term. One of the explanations given by Chua *et al.* (2009) is that utility functions that exhibit higher levels of loss aversion are able to reduce correlation asymmetry and provide greater downside diversification and upside unification. Therefore, a less conservative investor might execute trades that can appease a greater set of consenting measures (as shown by the indices of acceptability) but suffer economically. However, using asymmetric returns estimates are able to combine the objectives of both preferences in a successful manner across a variety of utility functions and data sets.

7. Conclusion

In asset allocation and funds management applications, Kritzman (2011) recommends the use of full-scale optimisation (FSO) as a suitable alternative to mean-variance (MV) optimisation to account for non-normal returns distributions and investors' preferences for higher moments. Although several papers show that FSO demonstrates enhanced performance benefits for investors with higher-moment preferences compared to MV optimisation

(Adler and Kritzman, 2007), estimation error continues to persist in out-of-sample applications when historical asset returns are applied (Hagströmer *et al.*, 2008; Hagströmer and Binner, 2009).

Our work applies the Clayton canonical vine copula (CVC) to model asymmetric dependence across a portfolio of assets. By summating the Clayton copula parameter across the vine model consisting of US industry asset returns, we create a US Asymmetric Dependence Index that shows sharp increases in correlation asymmetry during notable crisis events in US financial markets. Based on these findings, in a tactical asset allocation exercise using FSO, we reduce estimation error by exploiting the persistence of distributional asymmetries in estimation of expected returns. More specifically, we characterise asymmetries in asset correlations, volatility and residuals within the marginal distributions via the Clayton CVC (Aas *et al.*, 2009), GARCH-GJR (Glosten *et al.*, 1993) and skewed Student *t* (Skew-T) (Hansen, 1994), respectively, in an out-of-sample study. Our sample period spans several decades across portfolios of international country and US industry indices. We explore the kinked power, bilinear and S-curve (Kahneman and Tversky, 1979) utility functions for a range of risk parameters to capture behavioural biases that are increasingly evident in investors (Benson *et al.*, 2007; Gerrans *et al.*, 2015; Hoffmann and Post, 2015). We evaluate the performance benefits of estimating returns asymmetries using the indices of acceptability (Cherny and Madan, 2009) to provide an indication of the consensus regarding the direction of trades, and a range of risk-adjusted return metrics to provide economic intuition.

Thus, our work builds upon studies by Hatherley and Alcock (2007) and Durand *et al.* (2010) who have applied copulas in asset allocation and risk management analysis, respectively. Hatherley and Alcock (2007) use a Clayton copula to forecast estimates of expected returns in minimising conditional value-at-risk (CVaR) for a small portfolio of three assets on the Australian Stock Exchange (ASX). Durand *et al.* (2010) derive a copula that exhibits features of both the Frank and Gumbel copulas to examine the flight-to-quality effect between bonds and equities.

We find that when historical returns are applied, the less conservative investors (i.e. S-curve investors) generate poorer economic outcomes with Sharpe ratios between −4.11 and 1.37 compared to the more cautious (i.e. kinked power investors) with Sharpe ratios between 1.07 and 8.27. However, less conservative investors who are more risk-seeking, with a preference for positive skewness, score higher on the indices of acceptability. As measured by AIMAXMIN, the S-curve investors have a range of 2.72 to 4.91 and the kinked power investors have a range of 2.50 to 3.52. Application of asymmetric returns improves performance outcomes as measured by risk-adjusted returns. Investors with S-curve preferences have the highest economic benefits from using asymmetric returns estimates compared to the more loss-averse investors with kinked power and bilinear utility functions. We find that S-curve investors have a Sharpe ratio range of −4.11 to 1.37 using historical samples that improves to 9.57 to 14.91 with asymmetric estimates.

Chua *et al.* (2009) find that optimising a bilinear utility function results in more enhanced performance outcomes due to the reduction in correlation asymmetries within the portfolio. Our investigation supports their analysis as we find that loss-averse investors result in more successful economic outcomes in the long term based on the use of historical returns samples. We find that asymmetric returns estimates largely benefit investors who exhibit low levels of loss aversion and are more risk-seeking in their approach to investments (i.e. S-curve). Intuitively, estimation models that incorporate returns asymmetries are analogous to an investor having a conservative view about future returns. Therefore, there are reduced benefits to an investor who has a loss-averse approach to portfolio optimisation (i.e. bilinear, kinked power) as any correlation asymmetries that exist have already been exploited in the estimation process. Therefore, if one is unable to account for returns asymmetries due to the sophistication of the mathematical modelling required, the best approach to achieve a reasonably successful economic outcome in the long run is to exercise a degree of caution when investing. Otherwise, if an investor has the capability to model correlation asymmetries in their estimation process, they are able to improve the performance outcomes for a wide variety of investor preferences.

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Appendix I. Dependence model (vine copula)

Conceptually, a copula is a multivariate distribution that combines two (or more) given marginal distributions into a single joint distribution.[14] Archimedean copula models are commonly used as they may incorporate flexible range of dependence structures. However, they consist of one-parameter or two-parameter models of the dependence structure regardless of the number of assets. This might be sufficient for a model of two or three assets, but more complex models will likely require more flexible parameterisation.

Flexibly modelling dependence is straightforward for bivariate data but is far more difficult for higher dimensions as the choice of copulas then becomes limited to Gaussian or Student t copulas that only capture elliptical dependence. The Gaussian copula lacks tail dependence, and even though the multivariate Student t copula is able to generate different tail dependence[15] for each pair of variables, it is restricted to have the same upper and lower tail dependence.

The work of Bedford and Cooke (2002) and that of Aas *et al.* (2009) have led to the development of flexible and scalable copula models, also known as the canonical vine copula (CVC). The CVC allows us to overcome the limitations of traditional copula models by modelling dependence using simple local building blocks (pair-copulas) based on conditional independence.

A joint probability density function of n variables $u_1, u_2, \ldots, u_n$ can be decomposed without loss of generality by iteratively conditioning where

$$f(u_1, u_2, \ldots, u_n) = f(u_1) \cdot f(u_2|u_1) \cdot f(u_3|u_1, u_2) \cdots f(u_n|u_1, \ldots, u_{n-1}). \qquad (I.1)$$

Each of the factors in this product can be decomposed further using conditional copulas. For example, the first conditional density can be decomposed into the copula function c_{12} (the copula linking u_1 and u_2) multiplied by the density of u_2 such that

$$f(u_2|u_1) = c_{12}[F_1(u_1), F_2(u_2)]f_2(u_2), \qquad (I.2)$$

where $F_i(.)$ is the cumulative distribution function (cdf) of u_i. The joint density of the three-dimensional case can be decomposed in a hierarchal construction

[14] Nelsen (2006) and Joe (1997) both provide an excellent introduction to copula theory.

[15] Tail dependence for a multivariate Student t copula is a function of the correlation and degrees of freedom.

based on pair-copulas with conditional cdf as arguments, and marginal densities as

$$f(u_1, u_2, u_3) = c_{23|1}(F_{2|1}(u_2|u_1), F_{3|1}(u_3|u_1); u_1)c_{12}(F_1(u_1), F_2(u_2)) \\ c_{13}(F_1(u_1), F_3(u_3))f_1(u_1)f_2(u_2)f_3(u_3). \tag{I.3}$$

We assume that the pair-copulas are independent of the conditioning variables, except through the conditional distributions as shown in Equation (I.4). Hobæk Haff *et al.* (2010) show that this practical approximation remains reasonably accurate and several benefits such as improved efficiency, flexibility and robustness of the model inferencing process apply.

$$f(u_1, u_2, u_3) = c_{23|1}(F_{2|1}(u_2|u_1), F_{3|1}(u_3|u_1))c_{12}(F_1(u_1), F_2(u_2)) \\ c_{13}(F_1(u_1), F_3(u_3))f_1(u_1)f_2(u_2)f_3(u_3). \tag{I.4}$$

Joe (1997) proves that conditional distribution functions can be solved using

$$F(u|v) = \frac{\partial C_{u,v_j|v_{-j}}\left(F\left(u|v_{-j}, F\left(v_j|v_{-j}\right)\right)\right)}{\partial F\left(v_j|v_{-j}\right)}, \tag{I.5}$$

where v_{-j} is the vector v that excludes the component v_j. The above example is conditioned upon y_1.

Appendix II. Marginals modelling

We apply an AR(2) model for the mean equation and capture asymmetric volatility using the GARCH-GJR model (Glosten *et al.*, 1993). The impact of skewness and kurtosis within the residuals (error distribution) is modelled using the skewed Student t (Skew-T) set-up of Hansen (1994) to incorporate the effects of skewness and kurtosis. There will be a higher probability of a large number of negative returns than positive returns during bear markets. Therefore, these effects are captured by a negative λ that indicates a left-skewed density. Thus, our marginal model is given by:

$$y_{i,t} = c_i + \sum_{j=1}^{2} \phi_{i,j} \cdot y_{i,t-j} + \sqrt{h_{i,t}} \cdot z_{i,t}, \quad \text{for } i = 1, \ldots, N, \tag{II.1}$$

$$h_{i,t} = \omega_i + \alpha_i y_{i,t-1}^2 + \beta_i h_{i,t-1} + \gamma_i y_{i,t-1}^2 I_{i,t-1}, \tag{II.2}$$

$$z_{i,t} \sim \text{skewed Student } t(v_i, \lambda_i), \tag{II.3}$$

where $I_{i,t-1} = 0$ if $y_{i,t} \geq 0$ and $I_{i,t-1} = 1$ if $y_{i,t} < 0$. The skewed Student t density is given by

$$g(z|v,\lambda) = \begin{cases} bc\left(1+\frac{1}{v-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-(v+1)/2} & z < -a/b, \\ bc\left(1+\frac{1}{v-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-(v+1)/2} & z \geq -a/b \end{cases}. \tag{II.4}$$

The constants a, b and c are defined as follows:

$$a = 4\lambda c\left(\frac{v-2}{v-1}\right), b^2 = 1+3\lambda^2-a^2, c = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi(v-2)}\Gamma\left(\frac{v}{2}\right)}. \tag{II.5}$$

Appendix III. Full-scale optimisation

Full-scale optimisation identifies the optimal portfolio given a set of return distributions and a utility function outlining investor preferences. It implicitly takes into account all of the features of the empirical sample including skewness, kurtosis and any other peculiarities of the distribution and calculates the portfolio weights that maximise a given utility function given by (Equation III.1).

$$w^{FSO} = \arg\max_{w}\left(\frac{1}{T}\sum_{t=1}^{T} U(w'R_t)\right), \tag{III.1}$$

subject to

$$w'\mathbf{1}_N = 1, \tag{III.2}$$

$$-1 \leq w \leq 1. \tag{III.3}$$

The objective is to maximise the utility function U using the vector w that contains the weights for a portfolio of N assets. R is a matrix of $N \times T$, where T is the set of sample (or expected) returns available for each asset. Chronological order of these sample observations is ignored, and they are treated as future scenarios with equal probability. $\mathbf{1}_N$ denotes a vector of ones. Utility is evaluated for every combination of weights for all scenarios in the returns sample R. The optimal portfolio weights w^{FSO} are the weights w that produce the highest average utility over the entire sample of R returns.

Portfolio weights w can be subject to constraints as required by the user. In our application, we apply a budget constraint such that the sum of all portfolio weights must equal to unity and allow for short sales as shown in Equations (III.2) and (III.3), respectively.

As the utility surface is often non-convex, analytical solutions are infeasible; thus, grid searches might be used. However, grid searches are computationally intensive. For example, we apply a precision of 0.01 percent for each asset weight in a grid search for a portfolio of 17 assets, resulting in 4.85×10^{50} computations.[16] A suitable alternative is to use heuristic or deterministic global search algorithms.[17] Heuristic search algorithms such as genetic (Holland, 1975), simulated annealing (Kirkpatrick *et al.*, 1983), threshold accepting (Dueck and Scheuer, 1990) and differential evolution (Storn and Price, 1997) algorithms are popular as the inclusion of non-deterministic elements and acceptance of occasional impairments in the optimisation process allow local optima to be overcome easily at the cost of non-replicable results each time the optimisation process is restarted. Alternatively, the results from deterministic methods are replicable as they take steps from first-order conditions to find a suitable trajectory through the search space that leads to the optimum. Previous difficulties with deterministic search algorithms ending in the local rather than global optimum have now been overcome with modern parallel computing capacity.[18] We use the MultiStart solver that is the most efficient and robust global search algorithm in Matlab. We generate 50 random starting vectors of size N in a gradient search to solve for the global optimal portfolio weights across multiple processors in parallel. In our application of the algorithm, at least 80 percent or more of these vectors must converge upon the same global optimum to select the N optimal asset weights maximising the investor's utility function for the given returns sample R.

Appendix IV. Utility functions

The application of utility theory by Von Neumann *et al.* (1953) to portfolio management can be found as early as Tobin (1958). However, Levy (1969) and Samuelson (1970) discuss the relevance of higher moments for investment decisions, recognising that during realistic portfolio management scenarios, investors often express preferences that imply more complex utility functions

[16] Hagströmer *et al.* (2008) report the number of possible solutions as given by $m = \prod_{N=1}^{N-1} \frac{1/\delta+N}{N}$ where N is the number of assets and δ is the precision of the asset weights.

[17] For more information on the application of heuristic algorithms and deterministic search methods in finance, refer to Gilli *et al.* (2008) or Konno (2005), respectively.

[18] The size of the portfolios, asset weight precisions and length of our multiperiod study in our implementation of FSO are feasible due to the availability of the high-performance distributed network computing system at our home institution.

than power or quadratic utility (see Litterman (2003, Ch. 2) and Meucci (2005, Ch.5)). Investor's preferences for higher-order moments can be approximated using a Taylor series expansion around the mean, with investors showing positive (negative) signs on the derivative with respect to the odd (even) moments (Arditti, 1967; Scott and Horvath, 1980; Harvey *et al.*, 2010). Arditti (1971) shows supportive empirical evidence by documenting that although mutual funds might seem to exhibit poorer Sharpe ratios compared to the market index, they exhibit greater positive skewness. Arditti and Levy (1975) detail a method for generating a three-moment efficient frontier for multiperiod investments. They report that for investors with short investment horizons, skewness of portfolios can be ignored, but for longer horizons, the distributions' skewness can be significant and becomes an important variable in decision-making.

Bilinear and kinked power utility functions are investigated as both types are able to capture one of the central tenets in modern portfolio management, that is loss aversion. Both of these functional forms are characterised by a critical point of investment return where returns are given disproportionately low utility. The objective of limiting losses is motivated by monetary, regulatory requirements, and risk management. For example, an investor might require a minimum level of wealth to maintain a certain lifestyle that can change dramatically if this threshold is broken. A decline in wealth below a certain level could breach a loan covenant or even cause an investor to become insolvent.

Bilinear utility functions are formed by linear splines of different slopes on each side of a critical threshold point. As they consist of straight lines, aside from the kink at the critical point given by θ, bilinear functions do not reflect risk aversion in the same manner as power utility because the marginal utility does not decrease as returns increase. Returns are penalised with low utility if they are below the critical point θ and the higher the value of P, the greater the penalty applied. Risk-seeking behaviour is characterised as utility functions that have greater upside relative to downside preferences. For bilinear utility functions, investors with parameter $P = 1$ are more risk-seeking than $P = 3$, followed lastly by $P = 5$.

Kinked power utility functions incorporate loss aversion with λ that creates a kink at critical point θ as returns below it have a disproportionately lower weight. The degree of the investor's sensitivity to loss and relative risk aversion (RRA) is proportional to the values of λ and γ, respectively. For kinked power utility investors, those with parameters $\lambda = 3$; $\gamma = 1$ are more risk-seeking than $\lambda = 1$; $\gamma = 1$, followed lastly by $\lambda = 3$; $\gamma = 3$.

S-curve utility functions are important as proponents of behavioural finance have noted a number of contradictions to the neoclassical view of expected utility maximisation. Kahneman and Tversky (1979) find that investors focus on returns from an investment rather than wealth levels. In this regard, under prospect theory, investors exhibit risk aversion in the domain of gains but are risk-seeking in the domain of losses. Empirical studies find support for these behavioural effects in ex ante decision problems (Thaler *et al.*, 1997) and for dynamic portfolio revisions where investors tend to sell winners early and to hold on to losers too long (Odean, 1998). This behaviour is best captured by an S-curve utility function that features an inflection point given by θ. The closer (further) returns are to (from) the inflection point, the S-curve function implies higher (lower) absolute values of marginal utility. The parameters τ_1 and A respectively determine the shape and magnitude of the downside of the function, whereas τ_2 and B determine the upside characteristics in the same manner. Benartzi and Thaler (1995) report that the main determinant of portfolio choice under S-curve preferences are the loss aversion parameters A and B and the higher the ratio of B/A, the higher the risk chosen for the portfolio. The curvatures given by τ_1 and τ_2 are of secondary importance and have a minor influence on portfolio allocations. Intuitively, for the S-curve, utility is concave when returns are positive, but for negative returns it becomes convex, showing risk-seeking behaviour. Under such preferences, the investor exhibits the willingness to take on more risk when losses are made. Therefore, investors with S-curve utility functions are more risk-seeking than those with kinked power or bilinear utility preferences due to the lower penalty of utility for returns below the inflection/kink point. Investors where $B/A = 3$ are the most risk-seeking, followed by $B/A = 2$ and lastly $B/A = 1$.

Appendix V. Portfolio performance with economic (risk-adjusted returns) and consensus (indices of acceptability) measures

To evaluate the reduction in estimation error between using estimates of expected returns incorporating correlation asymmetries against historical returns, we report the performance of the portfolio strategies using a range of risk-adjusted metrics to provide economic intuition, and the indices of acceptability (Cherny and Madan, 2009) that measure the degree of consent for the strategy. As our portfolio strategies optimise the investor's utility function directly, it is vital that we report a range of metrics to broadly capture investor preferences.

Using a range of risk-adjusted returns metrics (e.g. Sharpe ratio, Sortino ratio, Omega, Mean/CVaR) allows the measurement of the out-of-sample performance of dynamically adjusted portfolios where variance, downside and tail risk are captured. These metrics are commonly reported in the hedge fund literature as hedge funds typically generate non-Gaussian outcomes (Adler and Kritzman, 2007), require accurate identification of changes in exposure to risk factors for accurate evaluation of performance and exhibit increased exposure to downside risk (Pinnuck, 2004; Gerrans, 2006).

Indices of acceptability[19] are designed to convey an understanding of the set of personalised pricing kernels that view a trade with positive marginal value. Intuitively, as portfolio managers act as agents on behalf of investors,

[19] Details regarding the derivation of the indices of acceptability with concave distortion functions are given in Appendix VI.

personalised utility functions could be inappropriate. Therefore, a portfolio evaluation measure that captures the preferences of a wide range of investors and agents who constitute the market is desirable. As stated by Cherny and Madan (2009, p. 2600), 'the higher the level of acceptability, the greater is the set of consenting measures and the more likely that it is viewed positively by investors who are not at hand at the decision-making point'. Thus, the indices of acceptability can be used to gauge the degree of positive consensus surrounding the direction of the optimal trade. The higher the index value, the more agreeable the investor is with the direction of the trade(s).

Reporting both economic and consensus measures allows for two key insights: (i) an understanding of the utility functions, when optimised in a portfolio management scenario with historical returns, that produce the highest outcomes for the indices of acceptability (consensus measure) or risk-adjusted metrics (economic measure); and (ii) the utility functions that exhibit the greatest benefits from using asymmetric returns estimates.

As the indices of acceptability (Cherny and Madan, 2009) successfully capture the preferences of a broad proportion of market participants, we can observe each category of utility functions and their risk parameters to understand the type of behaviour investors should ideally exhibit to maximise the consensus regarding the direction of the trades. Cherny and Madan (2009) state that the evaluation of trades consists of the direction and scale of the trade. However, the scale of trade is affected by several issues that are personalised (e.g. level of personal risk aversion, wealth and borrowing ability of the individual trading) and market-based (e.g. depth of the market and the resulting impact of the trade on the terms of the trade). Therefore the direction of a trade is presumed to be a more objective consideration. Following this argument, an acceptability index is an indicator of the size of support (degree of consensus) provided to a marginal trade in the preferred direction.

This is important as it conceptualises the type of utility functions that optimise trades in the direction that a large proportion of investors in the market prefer. First, by analysing the results of utility function optimisations that consistently produce high values of the indices of acceptability, this would lead us to potentially conclude that most market participants would support those particular utility functions. Second, we can understand if a difference exists between the magnitude of the consensus (as defined the indices of acceptability) and economic motivations (as defined by the risk-adjusted return measures) for an investor.

For example, an unsophisticated fund manager trading on behalf of an investor would use historical returns samples as an estimate of expected returns and execute a series of trades in the preferred direction. If those trades result in a high index of acceptability and high risk-adjusted return, we can conclude that the fund manager's behaviour is ideal as he is executing trades that would receive a high level of support from the investors and he is economically successful. However, other outcomes might result such that the fund manager may score a high index of acceptability and a low economic outcome or the opposite.

Therefore, it is important to observe the economic results when historical returns are applied, as obtaining unadjusted historical data is a relatively simple process compared to producing asymmetric estimates where substantial mathematical complexity and computational resources are required. Therefore, we wish to observe the type of investors who exhibit the greatest benefits from applying asymmetric estimates as it is possible that certain types of investors (e.g. investors with a high degree of loss aversion) may not experience any performance enhancements.

Appendix VI. Indices of acceptability and concave distortion functions

Indices of acceptability are designed to convey an understanding of the set of personalised pricing kernels that view a trade with positive marginal value. High levels of acceptability translate to a larger pool of pricing kernels consenting to the trade. As provision of a specific utility function is not required, this effectively depersonalises the portfolio selection process and is therefore more closely related to the intuitions embedded in classical economics. Therefore, the indices of acceptability are ideal as a set of portfolio performance ranking measures.

The indices of acceptability are computed by inducing a hypothetical shock to portfolio returns with a concave distortion function to generate a stressed sample. The portfolio return X is modelled as a random cash flow with end of period distribution function F_X (X). The portfolio return is regarded as acceptable at a given level δ if the following condition as shown in Equation (VI.1) is satisfied.

$$E(\delta, X) \geq 0 \quad \text{where} \quad E(\delta, X) = \int_{-\infty}^{\infty} x d(\Psi_\delta(F_X(x))). \tag{VI.1}$$

Where Ψ_δ $(F_{X)}$ is a distortion function that is parameterised by distortion value δ.[20] In the special case that Ψ_δ $(F_X) = F_X$ (X), then $E(\delta,$ $X)$ in Equation (VI.1) is the expected value of X. However, if the distortion function Ψ_δ (F_X) is concave, losses are reweighted upwards when F_X (X) is close to zero, and gains are discounted when $F_X(X)$ is close to unity. Intuitively, this is consistent with the behaviour of risk-averse agents. Cherny and Madan (2009) consider four different concave distortion functions that result in the AIMIN, AIMAX, AIMINMAX, and AIMAXMIN measures.

[20] As performed by Cherny and Madan (2009), we use a distortion value of 2. Intuitively, the distortion value can be understood as the amount of stress applied to portfolio returns, therefore distorting the original returns distribution.

The AIMIN measure generates a distorted sample on forming the expectation of the minimum of several draws from a returns series as shown in Equation (VI.2).

$$\Psi_\delta(x) = 1 - (1 - x)^{\delta+1}, \quad \delta \in R_+, \quad x \in [0, 1]. \qquad (VI.2)$$

The condition $E(\delta, X) \geq 0$ is equivalent to describing that the expectation computed using the minimum of $(\delta + 1)$ draws from the distribution of the portfolio return X must be positive to be deemed an acceptable investment return. Therefore, this measure demonstrates that using the worst case of a distorted sample generated from the portfolio distribution still produces an acceptable investment opportunity. Thus, the AIMIN measure amplifies large losses by a distortion factor $(\delta + 1)$ and discounts large gains to zero.[21] Intuitively, as large positive returns are nullified and large losses are exaggerated, the AIMIN measure is suitable for investors who are loss-averse. For example, an investor with a bilinear utility function where $P = 5$ represents an investor with a high degree of loss aversion; therefore, to gauge the acceptability of a distribution of portfolio returns, the AIMIN measure is a more suitable metric.

The AIMAX measure generates a distorted sample on forming the expectation of the maximum of several draws from a returns series as given in Equation (VI.3). Mathematically, the AIMAX measure discounts large gains by a distortion factor and amplifies large losses to negative infinity.

$$\Psi_\delta(y) = y^{\frac{1}{\delta_2+1}}, \qquad \delta_2 \in R_+, \quad y \in [0, 1]. \qquad (VI.3)$$

Intuitively, as unbounded large weights are applied upon large losses, and large gains suffer a slight distortion of $(\delta + 1)$, the level of positive gains have a greater effect on the AIMAX metric and is therefore more suitable for investors greater preferences for positive skewness. For example, for kinked power utility investors where $\gamma = 3$ and $\lambda = 1$ and $\gamma = 3$ and $\lambda = 3$, the AIMAX and AIMIN should be used as performance measures, respectively. This is because kinked power investors characterised by $\gamma = 3$, $\lambda = 1$ have greater upside preferences relative to downside preferences compared to power investors characterised by $\gamma = 3$, $\lambda = 3$.

For a balanced perspective, a suitable metric would simultaneously discount large gains to zero and amplify large losses to infinity. This is performed by the AIMAXMIN and AIMINMAX metrics that are given by Equations (VI.4) and (VI.5), respectively.

$$\Psi_\delta(y) = (1 - (1 - y^{\delta+1})^{\frac{1}{\delta+1}}, \qquad \delta \in R_+, \quad y \in [0, 1]. \qquad (VI.4)$$

[21] For further details, see Section 3.8 of Cherny and Madan (2009).

$$\Psi_\delta(y) = 1 - (1 - y^{\frac{1}{\delta+1}})^{\delta+1}, \qquad \delta \in R_+, \quad y \in [0,1]. \qquad (VI.5)$$

The AIMAXMIN (AIMINMAX) constructs a stressed sample using an AIMIN (AIMAX) perspective followed by an AIMAX (AIMIN) perspective. Therefore, of the four indices of acceptability, the most generalised approach is given by the AIMAXMIN and AIMINMAX metrics and is applicable to a broad spectrum of market participants.

Appendix VII. International country data set

By performing portfolio optimisation upon several data sets with a broad coverage of assets across the entire US market or among the largest developed economies in the world, this improves the robustness of our investigation. Our international country data set spans from January 1970 to July 2010 (20 years of monthly data). It consists of market indices from the United States, Canada, Japan, France, Italy, Germany, Switzerland, the UK and Australia that is sourced from MSCI. Similar to our US data set, all indices in international country data set fail the Jarque–Bera test of normality at the 1 percent level.

7.1. International country analysis: Risk-adjusted returns

Table A6 shows a range of risk-adjusted metrics for the portfolio of nine international country indices. Generally, we find improvements for kinked power utility functions, less for bilinear, and none for S-curve utility when asymmetric returns estimates are applied.

In Panel A (kinked power utility), all cases show a greater Sharpe, Omega, Sortino and Mean/CVaR ratios when asymmetric estimates are applied. The largest improvements are demonstrated by $K_{3,3}$, namely the most conservative of all kinked power utility functions applied due to the higher degree of RRA and loss aversion. In contrast, the case of $K_{3,1}$, where investor's preferences exhibit lower degrees of loss aversion and higher RRA, produces the least improvement but has the highest value of risk-adjusted returns whether historical or asymmetric returns estimates are applied. When asymmetric estimates are applied, risk-adjusted return metrics are improved such that similar values are obtained across all kinked power utility functions.

In Panel B (bilinear utility functions), improvements across all risk-adjusted metrics are observed when asymmetric estimates are used for the cases of B_5 where $\theta_{-2\text{ percent}}$ and $\theta_{-5\text{ percent}}$, and B_3 where $\theta_{-5\text{ percent}}$. Under these utility function scenarios, the higher the degree of loss aversion, the greater the improvement when asymmetric estimates are applied. Across all values of θ, B_1 experiences large adverse effects when asymmetric estimates are applied as negative mean returns are produced. We find that the location of the kink has a

Table A6
Risk-adjusted return metrics for FSO utility functions - international country indices

Utility parameters		Sharpe ratio		Omega ratio		Sortino ratio		$\frac{\text{Mean}}{\text{CVaR}}$	
θ (%)	Risk parameters	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.
Panel A: Kinked power utility									
0	$K_{1,3}$	9.34	9.72	1.28	1.29	13.19	14.21	2.83	3.04
	$K_{3,3}$	7.18	10.34	1.21	1.32	9.88	15.34	2.22	3.26
	$K_{3,1}$	9.90	11.68	1.29	1.35	14.74	17.23	3.51	3.72
−2	$K_{1,3}$	8.33	10.36	1.25	1.31	11.78	15.37	2.61	3.34
	$K_{3,3}$	5.84	12.34	1.17	1.37	7.98	18.57	1.77	4.19
	$K_{3,1}$	9.90	11.68	1.29	1.35	14.74	17.23	3.51	3.72
−5	$K_{1,3}$	5.54	11.60	1.15	1.34	7.77	17.36	1.92	3.66
	$K_{3,3}$	2.01	12.63	1.05	1.39	2.74	19.38	0.66	4.10
	$K_{3,1}$	9.90	11.68	1.29	1.35	14.74	17.23	3.51	3.72
Panel B: Bilinear utility									
0	B_1	4.03	3.11	1.11	0.92	5.74	4.14	1.27	0.97
	B_3	11.05	9.04	1.35	1.26	16.66	13.07	3.49	2.78
	B_5	11.16	9.53	1.35	1.28	15.84	13.84	3.28	2.97
−2	B_1	4.02	3.13	1.11	0.92	5.71	4.16	1.27	0.97
	B_3	10.53	10.26	1.33	1.30	15.62	15.14	3.40	3.28
	B_5	8.71	10.59	1.26	1.32	12.10	15.74	2.62	3.46
−5	B_1	4.17	3.01	1.12	0.92	5.94	4.00	1.32	0.93
	B_3	6.65	10.42	1.19	1.30	9.51	15.30	2.32	3.21
	B_5	3.70	10.93	1.10	1.32	5.14	16.37	1.23	3.54
Panel C: S-curve utility									
0	S_3	3.82	5.05	1.10	1.14	5.49	7.23	1.33	1.83
	S_2	5.43	0.91	1.15	1.02	8.04	1.27	2.21	0.31
	S_1	9.05	3.56	1.26	1.10	13.39	4.83	3.17	1.03
−2	S_3	6.39	0.43	1.18	1.01	9.51	0.60	2.58	0.14
	S_2	7.44	1.30	1.21	1.04	10.93	1.75	2.82	0.35
	S_1	13.76	7.63	1.45	1.22	21.76	11.12	4.74	2.56
−5	S_3	9.90	5.74	1.30	1.17	14.65	7.98	3.10	1.48
	S_2	12.27	10.38	1.39	1.30	19.04	15.56	4.02	3.50
	S_1	11.14	12.82	1.36	1.39	16.91	19.84	3.31	4.37

This table shows the Sharpe, Omega, Sortino, and Mean/CVaR ratios for investors with kinked power (Panel A), bilinear (Panel B) and S-curve (Panel C) utility functions for a portfolio of 9 international country indices when historical samples or asymmetric estimates of expected returns are applied. The kink/inflection point of the utility function is denoted by θ.

greater impact on the risk-adjusted returns for bilinear utility compared to kinked power utility.

In Panel C (S-curve utility functions), the use of asymmetric estimates causes a deterioration in performance for most of the cases explored. Only small improvements are generated when $\theta_{0 \text{ percent}}$ and S_3, and $\theta_{-5 \text{ percent}}$ and S_1. We find that S_1 (S_3) where investors exhibit lower (high) levels of upside relative to downside preferences produces the highest (lowest) risk-adjusted returns across the different values of θ.

7.2. International country analysis: Indices of acceptability

Table A7 reports the indices of acceptability calculated upon the portfolio of nine international country indices. Although each utility function has different risk parameters, they all exhibit kink/inflection points, θ. A lower value of θ indicates lower risk aversion as this means that the kink/inflection point (where drops in utility occur) comes into effect in the negative returns region. Generally, we find that utility functions that exhibit low levels of loss aversion (e.g. B_1, $K_{3,1}$) and higher preferences for gains relative to losses (e.g. S_3) tend to have larger indices of acceptability.

Panel A reports results for the kinked power utility functions, where large values of γ and λ are indicative of an investor with higher RRA and loss aversion, respectively. We find that changes in risk parameters γ and λ have greater effects on the portfolios than do changes in θ. When investors have higher levels of RRA and lower levels of loss aversion (i.e. $K_{3,1}$), larger indices of acceptability are produced. A combination of $\theta_{0 \text{ percent}}$ and $K_{1,3}$ indicates an investor whose utility only increases moderately to changes in upside gain and is highly averse to losses. In this case, we find that asymmetric estimates produce better outcomes reflected by the larger values for AIMIN and AIMAXMIN compared to historical data. The AIMIN is a more suitable metric than the AIMAX for investors who are more loss-averse. Notably, even when the AIMAXMIN is applied, a more robust metric that also penalises large positive returns, the superior performance of using asymmetric estimates continues to persist.

In Panel B, where investors are assumed to exhibit bilinear utility, higher values of P indicate larger levels of loss aversion. The change in utility on the right side of the kink points remains fixed for B_1, B_3 and B_5. We find that the indices of acceptability are affected more by the changes in risk parameters rather than the location of the kink. When $\theta_{0 \text{ percent}}$ and B_5, this is the most loss-averse model in the bilinear utility function category. Asymmetric estimates produce higher AIMIN and AIMINMAX values compared to historical data. As the AIMIN metric is most suitable for loss-averse investors compared to AIMAX, this shows that asymmetric estimates enhance the performance. This result persists for the AIMINMAX metric that combines the desirable attributes of the AIMAX metric with the AIMIN metric. For $\theta_{-2 \text{ percent}}$ and $\theta_{-5 \text{ percent}}$, both B_3 and B_5 have similar values of indices of acceptability and all the indices

Table A7
Indices of acceptability results for FSO utility functions -international country indices

Utility parameters		AIMIN		AIMAX		AIMINMAX		AIMAXMIN	
θ (%)	Risk parameters	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.	Hist. sample	Asymm. est.
Panel A: Kinked power utility									
0	$K_{1,3}$	5.95	6.54	17.88	17.76	25.41	25.17	2.11	2.33
	$K_{3,3}$	5.44	6.97	17.08	18.11	24.30	25.63	1.91	2.50
	$K_{3,1}$	9.30	8.79	20.50	20.75	28.83	29.27	3.43	3.20
−2	$K_{1,3}$	6.10	6.75	17.66	18.00	25.07	25.50	2.17	2.41
	$K_{3,3}$	5.48	7.30	16.84	18.75	23.95	26.54	1.93	2.62
	$K_{3,1}$	9.30	8.79	20.50	20.75	28.83	29.27	3.43	3.20
−5	$K_{1,3}$	6.93	7.56	18.20	18.93	25.78	26.77	2.48	2.71
	$K_{3,3}$	5.83	7.47	16.89	18.49	23.98	26.13	2.06	2.69
	$K_{3,1}$	9.30	8.79	20.50	20.75	28.83	29.27	3.43	3.20
Panel B: Bilinear utility									
0	B_1	11.79	9.68	21.59	19.28	30.04	26.98	4.55	3.64
	B_3	6.51	6.49	18.15	17.95	25.74	25.46	2.34	2.31
	B_5	5.86	6.41	18.14	17.82	25.80	25.28	2.07	2.28
−2	B_1	11.78	9.68	21.63	19.28	30.10	26.99	4.54	3.64
	B_3	6.69	6.85	18.08	18.27	25.62	25.89	2.40	2.45
	B_5	5.79	6.66	17.31	17.93	24.60	25.40	2.04	2.38
−5	B_1	11.78	9.66	21.65	19.26	30.14	26.96	4.54	3.63
	B_3	7.48	7.59	18.64	19.20	26.34	27.16	2.70	2.73
	B_5	6.22	7.21	17.44	18.31	24.74	25.90	2.21	2.58
Panel C: S-curve utility									
0	S_3	13.74	12.10	23.10	21.45	31.94	29.81	5.43	4.64
	S_2	13.21	11.13	22.81	20.06	31.61	27.90	5.19	4.27
	S_1	11.02	8.88	21.70	19.87	30.37	27.97	4.13	3.26
−2	S_3	12.41	10.82	22.30	19.73	31.00	27.45	4.80	4.16
	S_2	11.71	9.86	22.41	20.29	31.29	28.44	4.46	3.69
	S_1	8.30	7.01	19.80	18.46	27.93	26.14	3.04	2.52
−5	S_3	9.21	8.98	20.19	19.65	28.39	27.63	3.40	3.31
	S_2	8.50	7.97	19.69	19.27	27.74	27.21	3.13	2.88
	S_1	7.38	7.35	19.08	18.79	26.98	26.58	2.68	2.65

This table shows the indices of acceptability metrics of AIMIN, AIMAX, AIMINMAX, and AIMAXMIN for investors with kinked power (Panel A), bilinear (Panel B) and S-curve (Panel C) utility functions a portfolio of 9 international country indices when historical samples or asymmetric estimates of expected returns are applied. The kink/inflection point of the utility function is denoted by θ.

are larger when asymmetric estimates are used rather than historical. Thus, we find that investors who exhibit larger degrees of loss-averseness exhibit greater benefits when asymmetric estimates are used.

In Panel C, S-curve utility functions tend to exhibit larger values for the indices of acceptability compared to kinked power and bilinear utility functions. However, there are no gains to be found when asymmetric estimates are used instead of historical data. This might be due to the characteristics of the international country portfolio and the size of the data set being too small for the CVC model to produce any potential enhancements. In addition, agents exhibiting S-curve preferences generally do not exhibit large degrees of loss aversion compared to bilinear and kinked power utility. We find that S-curve functions with S_3 exhibit the highest levels of the indices of acceptability. This outcome is similar to kinked power utility functions where $K_{3,1}$ as investors in these cases have a greater preference for upside gains and lower levels of loss aversion.